An Application of the Finite Element Method to Curve Fitting

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ABSTRACT

An application of the Finite Element Method (FEM) to the solution of a geometric problem is shown. The problem is related to curve fitting i.e. pass a curve trough a set of given points even if they are irregularly spaced. Situations where curves with cusps can be encountered in the practice and therefore smooth interpolatting curves may be unsuitable. In this paper the possibilities of the FEM to deal with this type of problems are shown. A particular example of application to road planning is discussed. In this case the funcional to be minimized should express the unpleasent effects of the road traveller. Some comparative numerical examples are also given.

INTRODUCTION

The main purpose of this paper is to show the possibility of using other curves than the traditional ones (straight lines, circles and clothoides) in order to define the longitudinal axis of a road. The paper has an interdisciplinary character and the proposed curves are simply the interpolation funcionts of the FEM and they are specified by minimizing a given functional.

The finite element method was first used more than twenty five years ago [1] in the solving of a structural problem. Since then it has been developed to a considerable degree to the point that now it constitutes an important tool for the studying of problems in the field of mathematics, enabling a large variety of physical situations to be dealt with. The mathematical foundations of the method are well established [2] , and numerous variations and formulations are possible: weighting function techniques (Galerkin, collocation, etc.), semianalytical procedures (finite strips, layers and finite prisms, etc.), as the well known boundary element method, are just some of the examples of the numerous possibilities which exist. See [3] for a summary of such examples. In the present work, a concrete application is described, namely applying the method in the planning of roads.

FORMULATING THE PROBLEM

From a mathematical point of view the horizontal proyection of the axis of the road can be established as follows

The coordinates of N points, $P_i(x_i,y_i)$, are given with reference to a global coordinate system x,y (figure 1). These points are ordered according to the forward direction of the axis $(i = 1,2,3,\dots N)$.

A continuous curve with continuous slope and curvature which passes through the above-mentioned points needs to be found. This curve may need to satisfy in addition a number of other "boundary" conditions, in such a way that it commands an entry slope, exit slope or slope at a midway point, or values of the curvatur s at some arbitrary points P_i.

APPLICATION OF THE FINITE ELEMENT MEHTOD. (FEM)

The segment i is defined by the two extreme points P_i and P_{i+1}, and, according to Figure 1, the following can be written:

$$l_i = \frac{1}{2}\sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}$$

$$\cos\alpha_i = \frac{x_{i+1}-x_i}{2l_i}\ ; \quad \sin\alpha_i = \frac{y_{i+1}-y_i}{2l_i}$$

The required curve $y = y(x)$ which passes through the points P_i and is continous C^2 will be restricted by the following "smoothing" condition:

$$y(x) \text{ minimizes the functional } \oint \frac{ds}{R^2} \qquad (1)$$

Where the curvilinear integral extends to $y = y(x)$, ds is the differential of arc and R the radious of curvature. The functional (1) to be minimized represents only a possibility among several ones. Other functionals can include higher order derivatives (first order derivative of the curvature) expressing the unpleasant effects of the road in the traveller. The mathematical treatment of these functionals by the FEM is similar to the one given here. See [4].

As it is known the finite element technique enables problem (1) to be solved by expressing the solution $y = y(x)$ as a sum of piece-wise functions. In this case the following considerations are valid:

For the segment i, $\overline{P_iP_{i+1}}$, the adimensional local coordinates (ξ,η), shown in Figure 2, are adopted and related to the global coordinates by means the following expressions:

$$\xi = \frac{1}{l_i}\left(x - \frac{x_i + x_{i+1}}{2}\right)\cos\alpha_i + \frac{1}{l_i}\left(y - \frac{y_i + y_{i+1}}{2}\right)\sin\alpha_i \qquad (2a)$$

$$\eta = -\frac{1}{l_i}\left(x - \frac{x_i + x_{i+1}}{2}\right)\sin\alpha_i + \frac{1}{l_i}\left(y - \frac{y_i + {}_{i+1}}{2}\right)\cos\alpha_i \qquad (2b)$$

the inverse transformation of which is:

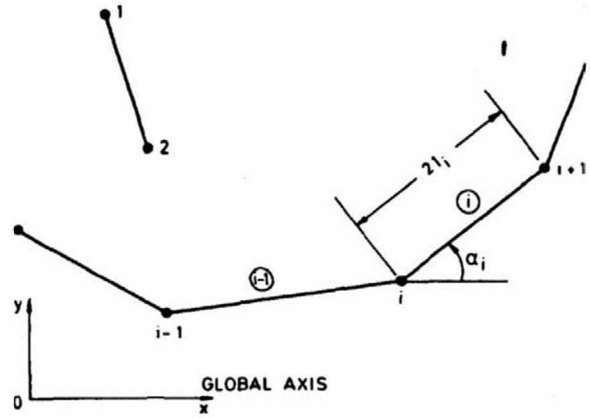

Figure 1. Polygon

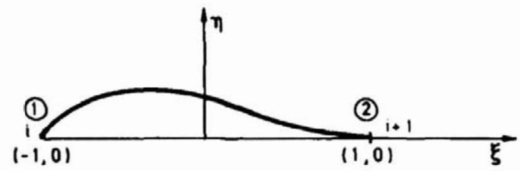

Figure 2. Generic element i

$$x = \frac{x_i + x_{i+1}}{2} + l_i(\xi\cos\alpha_i - \eta\sin\alpha_i) \tag{3a}$$

$$y = \frac{y_i + y_{i+1}}{2} + l_i(\xi\sin\alpha_i + \eta\cos\alpha_i) \tag{3b}$$

where $2l_i$ is the length of the segment i.

As is usual in the FEM the required curve $y = y(x)$ may be found within the section $\overline{P_iP_{i+1}}$ in the following form:

$$\eta = \{N_1, \bar{N}_1, N_2, \bar{N}_2\} \begin{vmatrix} \theta_1 \\ c_1 \\ \theta_2 \\ c_2 \end{vmatrix} \tag{4}$$

where θ_α, c_α $(\alpha = 1,2)$ are the slopes and curvatures of the point α (first or second) corresponding to i and i+1 respectively in the case of Figure 2. As can be noticed, a local numbering has been introduced in the segment $\overline{P_iP_{i+1}}$. The functions N_α, $\bar{N}_\alpha$ $(\alpha = 1,2)$ correspond to the interpolation functions or shape functions and they normally adopt polynomials of the abscissa ξ.

In order to linearize the problem, it is assumed that the points P_i and P_{i+1} are sufficiently close and then the curvature can be approximately expressed by the second derivative of the abscissa with respect to the ordinate, i.e. the following approximations is accepted:

$$\left(\frac{d\eta}{d\xi}\right)^2 << 1 \tag{5}$$

In this case the interpolation functions are fifth order hermitic polynomials:

$$N_1 = \frac{1}{16}(5-7\xi-6\xi^2+10\xi^3+\xi^4-3\xi^5) \tag{6a}$$

$$N_2 = -\frac{1}{16}(5+7\xi-6\xi^2-10\xi^3+\xi^4+3\xi^5) \tag{6b}$$

$$\bar{N}_1 = \frac{1}{16}(1-\xi-2\xi^2+2\xi^3+\xi^4-\xi^5) \tag{6c}$$

$$\bar{N}_2 = \frac{1}{16}(1+\xi-2\xi^2-2\xi^3+\xi^4+\xi^5) \tag{6d}$$

It can be shown that these interpolation functions satisfy the following equations:

$$N_1(-1) = 0; \quad N_1(1) = 0$$

$$\left.\frac{dN_1}{d\xi}\right|_{\xi=-1} = 1 \; ; \quad \left.\frac{dN_1}{d\xi}\right|_{\xi=1} = 0$$

$$\left.\frac{d^2N_1}{d\xi^2}\right|_{\xi=-1} = 0 \; ; \quad \left.\frac{d^2N_1}{d\xi^2}\right|_{\xi=1} = 0$$

$$N_2(-1) = 0 \; ; \quad N_2(1) = 0$$

$$\left.\frac{dN_2}{d\xi}\right|_{\xi=-1} = 0 \; ; \quad \left.\frac{dN_2}{d\xi}\right|_{\xi=1} = 1$$

$$\left.\frac{d^2N_2}{d\xi^2}\right|_{\xi=-1} = 0 \; ; \quad \left.\frac{d^2N_2}{d\xi^2}\right|_{\xi=1} = 0$$

$$\bar{N}_1(-1) = 0 \; ; \quad \bar{N}_1(1) = 0$$

$$\left.\frac{d\bar{N}_1}{d\xi}\right|_{\xi=-1} = 0 \; ; \quad \left.\frac{d\bar{N}_1}{d\xi}\right|_{\xi=1} = 0$$

$$\left.\frac{d^2\bar{N}_1}{d\xi^2}\right|_{\xi=-1} = 1 \; ; \quad \left.\frac{d^2\bar{N}_1}{d\xi^2}\right|_{\xi=1} = 0$$

$$\bar{N}_2(-1) = 0 \; ; \quad \bar{N}_2(1) = 0$$

$$\left.\frac{d\bar{N}_2}{d\xi}\right|_{\xi=-1} = 0 \; ; \quad \left.\frac{d\bar{N}_2}{d\xi}\right|_{\xi=1} = 0$$

$$\left.\frac{d^2\bar{N}_2}{d\xi^2}\right|_{\xi=-1} = 0 \; ; \quad \left.\frac{d^2\bar{N}_2}{d\xi^2}\right|_{\xi=1} = 1$$

These properties characterize the shape functions N_α, $\bar{N}_\alpha$, and have allowed its analytical determination. These functions are shown in Figure 3.

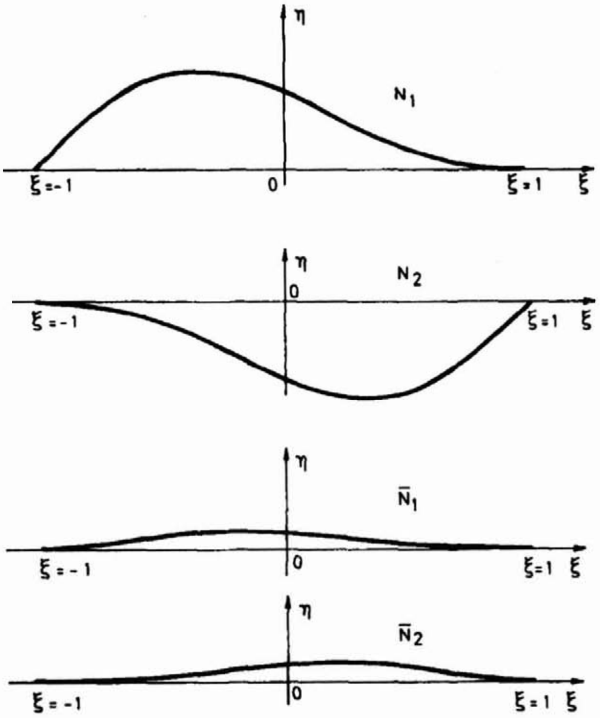

Figure 3. Shape functions.

Equation (4) can be writeen more conveniently in the following way:

$$\eta = \{\underline{N}_1^* \underline{N}_2^*\} \begin{vmatrix} \underline{d}_1 \\ \underline{d}_2 \end{vmatrix} \qquad (7a)$$

with:

$$\underline{N}_\alpha^* = \{N_\alpha \ \bar{N}_\alpha\} \qquad (7b)$$

$$\underline{d}_\alpha = \begin{vmatrix} \theta_\alpha \\ c_\alpha \end{vmatrix} = \begin{vmatrix} (\frac{d\eta}{d\xi})_\alpha \\ (\frac{d^2\eta}{d\xi^2})_\alpha \end{vmatrix} \ (\alpha = 1,2) \qquad (7c)$$

The continuity conditions which need to be satisfied by the curve $y = y(x)$ are of the type C^2, i.e. the first and second derivatives with respect to the same axes must be the same at the joints which are considered as the extremes of the adjacent elements. That is to say, according to Figure 4 the following can be written:

$$\theta_2^{i-1} - m_i = \theta_1^i + m_i = \lambda_i \quad \text{(first derivatives)} \qquad (8a)$$

$$\frac{1}{l_{i-1}} c_2^{i-1} = \frac{1}{l_i} c_1^i = \bar{\lambda}_i \quad \text{(second derivatives)} \qquad (8b)$$

where:

$$m_i = \operatorname{tg} \frac{\alpha_i - \alpha_{i-1}}{2} \qquad (8c)$$

θ_α^j, c_α^j are the slope and curvature of the extreme α of the section $j(\alpha = 1,2;\ j = i-1,i)$.

The parameters λ_α and $\bar{\lambda}_\alpha$ must be selected in such a way that the functional (1) is minimized, i.e.:

$$E = E(y) = \frac{1}{2} \int (\frac{d^2y}{dx^2})^2 \, ds = \frac{1}{2} \sum_{i=1}^{I} \frac{1}{l_i} \int_{-1}^{1} (\frac{d^2\eta^i}{d\xi^2})^2 d\xi \qquad (9)$$

where η^i corresponds to the ordinate at the interval P_iP_{i+1}, given by the expresion (4) and I is the number of sections (equal to N-1 in the case of an open polygon).

The contribution to (9) of a generic element i gives:

$$E = \frac{1}{2l_i} \int_{-1}^{1} (\frac{d^2\eta^i}{d\xi^2})^2 d =$$

$$= \frac{1}{2L_i} \int_{-1}^{1} \left[\{\underline{N}_1^{*\prime\prime} \underline{N}_2^{*\prime\prime}\} \begin{vmatrix} \underline{d}_1 \\ \underline{d}_2 \end{vmatrix} \right]^T \{\underline{N}_1^{*\prime\prime} \underline{N}_2^{*\prime\prime}\} \begin{vmatrix} \underline{d}_1 \\ \underline{d}_2 \end{vmatrix} d\xi$$

that is:

$$E_i = \frac{1}{2l_i} \{\underline{d}_1^T \underline{d}_2^T\} \begin{vmatrix} \underline{K}_{11} & \underline{K}_{12} \\ \underline{K}_{21} & \underline{K}_{22} \end{vmatrix} \begin{vmatrix} \underline{d}_1 \\ \underline{d}_2 \end{vmatrix}$$

with:

$$\underline{K}_{\alpha\beta} = \int_{-1}^{1} \begin{vmatrix} N''_\alpha N''_\beta & N''_\alpha N''_\beta \\ \bar{N}''_\alpha \bar{N}''_\beta & \bar{N}''_\alpha \bar{N}''_\beta \end{vmatrix} d\xi$$

matrix of dimension (2x2) and where the second derivative with respect to ξ is indicated by a double accent.

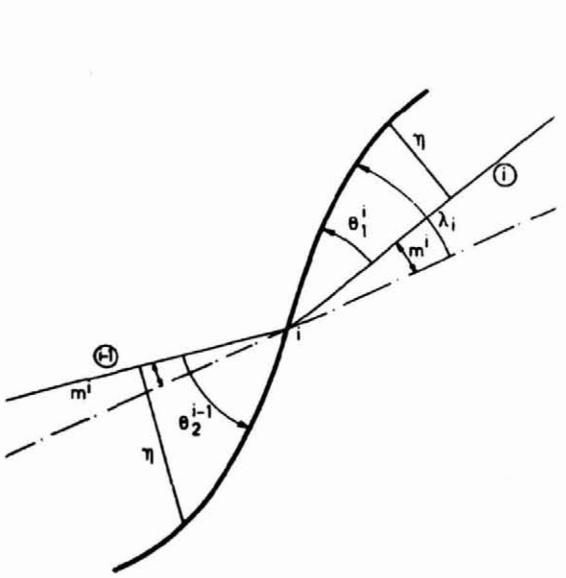

Figure 4. Continuity conditions.

The numerical expressions of the matrices $\underline{K}_{\alpha\beta}$ are:

$$\underline{K}_{11} = \frac{1}{35}\begin{vmatrix} 96 & 11 \\ 11 & 6 \end{vmatrix} \quad ; \quad \underline{K}_{12} = \frac{1}{35}\begin{vmatrix} 54 & -4 \\ 4 & 1 \end{vmatrix}$$

$$\underline{K}_{22} = \frac{1}{35}\begin{vmatrix} 96 & -11 \\ -11 & 6 \end{vmatrix} \quad ; \quad \underline{K}_{21} = \underline{K}_{12}^T$$

Provisionally it is assumed that the values m_i are equal to 0, i.e. The conditions (8) become:

$$\theta_2^{i-1} = \theta_1^i = \lambda_i$$

$$\frac{1}{L_{i-1}}\, c_2^{i-1} = \frac{1}{l_i}\, c_1^i = \bar{\lambda}_i$$

Consequently, the following expression is obtained:

$$E_i = \frac{1}{2}\,(\underline{u}_1^{i\,T}\ \underline{u}_2^{i\,T})\begin{vmatrix} \underline{K}_{11}^i & \underline{K}_{12}^i \\ \underline{K}_{21}^i & \underline{K}_{22}^i \end{vmatrix}\begin{vmatrix} \underline{u}_1^i \\ \underline{u}_2^i \end{vmatrix}$$

with:

$$\underline{K}_{\alpha\beta}^i = \frac{1}{l_i}\begin{vmatrix} \int_{-1}^{1} N''_\alpha\, N''_\beta\, d\xi & l_i \int_{-1}^{1} N''_\alpha\, \bar{N}''_\beta\, d\xi \\ l_i \int_{-1}^{1} \bar{N}''_\alpha\, N''_\beta\, d\xi & l_i^2 \int_{-1}^{1} \bar{N}''_\alpha\, \bar{N}''_\beta\, d\xi \end{vmatrix}$$

$\underline{u}^i = \begin{vmatrix} \lambda_\alpha \\ \bar{\lambda}_\alpha \end{vmatrix}$ of the extreme α of the section i, $(\alpha = 1,2)$

$$\underline{u}_1^i = \underline{u}_{i-1}$$

$$\underline{u}_2^i = \underline{u}_i$$

As a result the functional (9) becomes:

$$E = \sum_{i=1}^{I} E_i$$

the minimum of which is reached for the condition below:

$$\frac{\partial E}{\partial \underline{u}^i} = 0 \quad \text{for} \quad i = 1,2,\dots,\ N \text{ (joints)}$$

The following system is obtained:

$$\begin{vmatrix} \underline{K}_{11}^1 & \underline{K}_{12}^1 & 0 & \dots 0 & 0 \\ \underline{K}_{21}^1 & \underline{K}_{22}^1+\underline{K}_{11}^2 & \underline{K}_{12}^2 & \dots 0 & 0 \\ 0 & \underline{K}_{21}^2 & \underline{K}_{22}^2+\underline{K}_{11}^3 & \dots 0 & 0 \\ \vdots & \vdots & \vdots & \dots \vdots & \vdots \\ 0 & 0 & 0 & \dots \underline{K}_{22}^{I-1}+\underline{K}_{11}^I & \underline{K}_{12}^I \\ 0 & 0 & 0 & \underline{K}_{21}^I & \underline{K}_{22}^I \end{vmatrix}\begin{vmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \underline{u}_3 \\ \vdots \\ \underline{u}_I \\ \underline{u}_{I+1} \end{vmatrix} = \underline{0} \quad (10)$$

which corresponds to the situation of open polygon which ia a normal situation in road practice. $(N = I+1)$.

Obviously for the linear system of equations (10) with 2N unknowns there exists the trivial solution: $\underline{u}_i = 0$ for all i, which corresponds to a straight line.

In order to obtain the final solution, distortions are in troduced at each section i, with the following values at the extremes:

$$\hat{\theta}_1^i = -m_i \quad ; \quad \hat{\theta}_2^i = m_{i+1}$$

That means the "initial forces or equivalent nodal forces $\hat{p}_\alpha^i$" which appears in each section, and their expression is:

$$\begin{vmatrix} \hat{\underline{p}}_1^i \\ \hat{\underline{p}}_2^i \end{vmatrix} = \begin{vmatrix} \underline{K}_{11}^i & \underline{K}_{12}^i \\ \underline{K}_{21}^i & \underline{K}_{22}^i \end{vmatrix}\begin{vmatrix} \underline{u}_{10}^i \\ \underline{u}_{20}^i \end{vmatrix}$$

with:

$$\underline{u}_{10}^i = \begin{vmatrix} -m_i \\ 0 \end{vmatrix} \quad ; \quad \underline{u}_{20}^i = \begin{vmatrix} m_{i+1} \\ 0 \end{vmatrix}$$

and

$\hat{\underline{p}}_\alpha^i = \begin{vmatrix} \mu_\alpha^i \\ \bar{\mu}_\alpha^i \end{vmatrix}$ $(\alpha = 1,2)$ are the dual (static) quantities of the (kinematic variables) slope and curvature.

In this way the final system of linear equations, which enables the unknown $\underline{u}$ to be determined, is obtained:

$$\begin{vmatrix} \underline{K}_{11}^1 & \underline{K}_{12}^1 & 0 & 0 & \dots & \dots \\ \underline{K}_{21}^1 & \underline{K}_{22}^1+\underline{K}_{11}^2 & \underline{K}_{12}^2 & 0 & \dots & \dots \\ 0 & \underline{K}_{21}^2 & \underline{K}_{22}^2+\underline{K}_{11}^3 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \underline{K}_{22}^I \end{vmatrix}\begin{vmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \underline{u}_3 \\ \cdot \\ \underline{u}_{I+1} \end{vmatrix} = -\begin{vmatrix} \hat{\underline{p}}_1^1 \\ \hat{\underline{p}}_2^1+\hat{\underline{p}}_1^2 \\ \hat{\underline{p}}_2^2+\hat{\underline{p}}_1^3 \\ \cdot \\ \hat{\underline{p}}_2^I \end{vmatrix} \quad (11)$$

with $I = N-1$, the number of sections and N the number of joints of the polygon.

Once the system (11) has been solved, the values of the slope and curvature at the extremes of each section i, $(\theta_\alpha^i, c_\alpha^i)$, can be deducted. By means of the iterpolation equations (4) and the transformation equations (3), the coordinates (x,y) of the different points of the required curve $y = y(x)$ can also be deduced.

EXAMPLES OF APPLICATION

In order to test the approximation of the method, several simple cases have been studied.

The first six cases, the data correspond to the coordinates of four points P_i situated along a circle of radius $R = 100$ and equally spaced the angle $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ (Figure 5). In some cases, the values of the slopes and/or curvatures are also specified in the two exteme points P_1 and P_4 i.e. the entry and exit joints. The results obtained from the FEM are compared to the values of the given circle. The numerical sensibility of the FEM is studied by analyzing sucesive cases corresponding to different values of the angle $\alpha (\alpha = 5°,\ 10°,\ 15°,\ 20°,\ 25°$ and $30°)$. Also a case with different separation between consecutive points is analyzed, namely $\alpha_1 = 10°$, $\alpha_2 = 20°$ and $\alpha_3 = 30°$. In the Table I the results obtained in the different cases are summarized. From this table the results obtained from the FEM and the values of the given circle are compared. The practical concordance between the above values

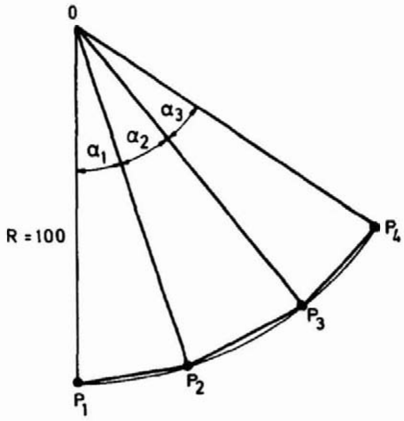

Figure 5. Example of application 1.

is reached when the slope and curvature are specified at the entry and exit points. This conclusion still is valid even for the largest separation angles (30°) and different angles (10°, 20° and 30°). If only the slope is specified in the entry and exit points the above concordance still holds but not too closely. However, if no values of slope and curvature are given at the extreme points of the axis, the difference between the results increases. The explanation for difference has to be found from the fact it may be possible to find a curve different to the circle passing through the four points (P_i; $i = 1,2,3,4$) that produce a smaller value to the functional (1) than the circle.

Another example corresponds to the one shown in the Figure 6. It is a transition curve composed by two clothoides of parameter $A = 100$ and length $L = 100$. The coordinates (x_i, y_i), slope (θ_i) and curvature (c_i) of the two points P_1 and P_4 are specified and only the coordinates at P_2 and P_3. The points P_1 and P_4 are the entry and exit points. In order to check the efficiency of the method two intermediate points are situated randomly along the transition curve. Then, five extreme cases have been estudied and each of them is defined by the distances L_2 and L_3 of the points P_2 and P_3 respectively.

The results and the comparison with the values obtained from the transition curve are shown in Table II.

TABLE 1. Comparative analysis or results.
Circle. (Figure 5).

NODE	ALIGNEMENT (slopes)					CURVATURE				
	CIRCLE	FEM1	FEM2	FEM3	FEM4	CIRCLE	FEM1	FEM2	FEM3	FEM4
CASE: $\alpha_1 = \alpha_2 = \alpha_3 = 5°$										
1	0.0000	0.0000	0.0000	0.0205	0.0262	0.0000	0.0100	0.0100	0.0100	0.0000
2	0.0875	0.0875	0.0875	0.0795	0.0787	0.0100	0.0100	0.0100	0.0109	0.0120
3	0.1763	0.1763	0.1763	0.1845	0.1853	0.0100	0.0100	0.0100	0.0109	0.0120
4	0.2680	0.2680	0.2680	0.2461	0.2401	0.0100	0.0100	0.0100	0.0100	0.0000
CASE: $\alpha_1 = \alpha_2 = \alpha_3 = 10°$										
1	0.0000	0.0000	0.0000	0.0410	0.0523	0.0100	0.0100	0.0100	0.0100	0.0000
2	0.1763	0.1763	0.1763	0.1601	0.1585	0.0100	0.0100	0.0100	0.0109	0.0120
3	0.3640	0.3640	0.3640	0.3819	0.3837	0.0100	0.0100	0.0100	0.0109	0.0120
4	0.5774	0.5774	0.5774	0.5239	0.5096	0.0100	0.0100	0.0100	0.0100	0.0120
CASE: $\alpha_1 = \alpha_2 = \alpha_3 = 15°$										
1	0.0000	0.0000	0.0000	0.0615	0.0784	0.0100	0.0100	0.0101	0.0100	0.0000
2	0.2680	0.2680	0.2680	0.2428	0.2403	0.0100	0.0101	0.0101	0.0110	0.0121
3	0.5774	0.5773	0.5773	0.6090	0.6123	0.0100	0.0101	0.0101	0.0110	0.0121
4	1.0000	1.0000	1.0000	0.8841	0.8545	0.0100	0.0100	0.0101	0.0100	0.0000
CASE: $\alpha_1 = \alpha_2 = \alpha_3 = 20°$										
1	0.0000	0.0000	0.0000	0.0918	0.1045	0.0100	0.0100	0.0102	0.0100	0.0000
2	0.3640	0.3641	0.3640	0.3290	0.3255	0.0100	0.0102	0.0102	0.0111	0.0122
3	0.8391	0.8390	0.8391	0.8921	0.8987	0.0100	0.0102	0.0102	0.0111	0.0122
4	1.7321	1.7321	1.7321	1.4450	1.3781	0.0100	0.0100	0.0102	0.0100	0.0000
CASE: $\alpha_1 = \alpha_2 = \alpha_3 = 25°$										
1	0.0000	0.0000	0.0000	0.1023	1.1305	0.0100	0.0100	0.0102	0.0100	0.0000
2	0.4663	0.4665	0.4663	0.4199	0.4154	0.0100	0.0103	0.0102	0.0112	0.0123
3	1.1918	0.1914	1.1918	1.2884	1.2984	0.0100	0.0103	0.0102	0.0112	0.0123
4	3.7321	3.7321	3.7321	2.6268	2.4223	0.0100	0.0100	0.0102	0.0100	0.0000
CASE: $\alpha_1 = \alpha_2 = \alpha_3 = 30°$										
1	0.0000	0.0000	0.0000	0.1226	0.1563	0.0100	0.0100	0.0104	0.0100	0.0000
2	0.5774	0.5777	0.5774	0.5175	0.5117	0.0100	0.0104	0.0104	0.0113	0.0124
3	1.7321	1.7310	1.7321	1.9272	1.9479	0.0100	0.0104	0.0104	0.0113	0.0124
4	-	-	-	8.1569	6.3987	0.0100	0.0100	0.0104	0.0100	0.0000
CASE: $\alpha_1 = 5°$; $\alpha_2 = 10°$; $\alpha_3 = 30°$										
1	0.0000	0.0000	0.0000	0.0447	0.0561	0.0100	0.0100	0.0100	0.0100	0.0000
2	0.1763	0.1760	0.1759	0.1524	0.1502	0.0100	0.0101	0.0101	0.0101	0.0107
3	0.5774	0.5755	0.5757	0.6229	0.6280	0.0100	0.0103	0.0102	0.0118	0.0320
4	1.7321	1.7321	1.7321	1.3456	1.2575	0.0100	0.0100	0.0104	0.0100	0.0000

FEM1: Specified values of the slopes and curvatures in the two extreme points.
FEM2: Specified values of the slopes in the two extreme points.
FEM3: Specified values of the curvatures in the two extreme points.
FEM4: Free values of the slopes and curvatures in all the points.

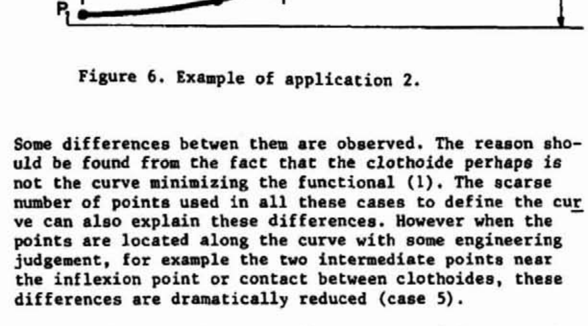

Figure 6. Example of application 2.

Some differences betwen them are observed. The reason should be found from the fact that the clothoide perhaps is not the curve minimizing the functional (1). The scarse number of points used in all these cases to define the curve can also explain these differences. However when the points are located along the curve with some engineering judgement, for example the two intermediate points near the inflexion point or contact between clothoides, these differences are dramatically reduced (case 5).

If the number of points to define the transition curve is increased to six (Figure 7), the results obtained are gi ven in the Table III and the increase in the accuracy is observed.

or directly a non linear mathematical programming thechnique in order to minimize the functional (1).

TABLE II. Comparative analysis of results. Clothoide, 4 points. (Figure 6).

	ALIGNMENT (slopes)									
	CASE1		CASE2		CASE3		CASE4		CASE5	
NODE	EXACT	FEM	EXACT	FEM	EXACT	FEM	EXACT	FEM	EXACT	FEM
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.2255	0.2605	0.0555	0.0605	0.0555	0.0852	0.0555	0.1093	0.4287	0.4461
3	0.2255	0.2605	0.2255	0.2021	0.0555	0.0857	0.2255	0.2131	0.4287	0.4461
4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

	CURVATURE									
	CASE1		CASE2		CASE3		CASE4		CASE5	
NODE	EXACT	FEM	EXACT	FEM	EXACT	FEM	EXACT	FEM	EXACT	FEM
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0067	0.0087	0.0033	0.0041	0.0033	0.0070	0.0033	0.0082	0.0090	0.0098
3	-0.0067	-0.0087	0.0067	0.0029	-0.0033	-0.0070	-0.0067	-0.0062	-0.0090	-0.0098
4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

CASE	6	7	8
L_1	0.00	0.00	0.00
L_2	33.33	66.66	33.33
L_3	66.67	90.00	90.00
L_4	133.33	110.00	110.00
L_5	166.67	133.33	166.67
L_6	200.00	200.00	200.00

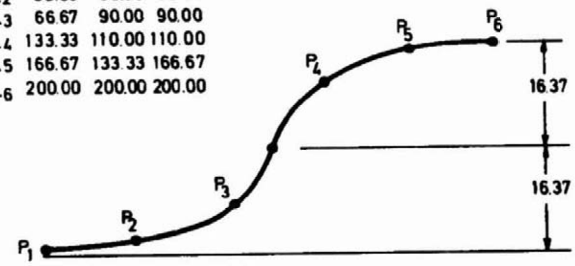

Figure 7. Example of application 3.

From the two above examples it can be deducted that the FEM allows to define unequivocally the curve of the road axis, subject to be continuous C^2 and minimizing a functional of the type (1). In this way, it is not ne cessary to use the traditional special curves such as straight lines, circles and clothoides, and therefore a more wide freedom for the road designer is reached. The axis can be defined using the FEM simply by the no de coordinates of some special points (joints) and so me extra constraints in the slope and/or curvature. The coordinates of intermediate points between two con secutive joints can be obtained from the use of the shape or interpolation functions (6).

TABLE III. Comparative analysis of results. Clothoide, 6 points. (Figure 7).

	ALIGNMENT (slopes)						CURVATURE					
	CASE6		CASE7		CASE8		CASE6		CASE7		CASE8	
NODE	EXACT	FEM	EXACT	FEM	EXACT	FEM	EXACT	FEM	EXACT	FEM	EXACT	FEM
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0,0000	0.0000	0.0000
2	0.0555	0.0491	0.2255	0.2113	0.0555	0.0529	0.0033	0.0026	0.0067	0.0065	0.0033	0.0030
3	0.2255	0.2526	0.4287	0.4423	0.4287	0.4451	0.0067	0.0094	0.0090	0.0106	0.0090	0.0100
4	0.0255	0.2526	0.4287	0.4423	0.4287	0.4451	-0.0067	-0.0094	-0.0090	-0.0106	-0.0090	-0.0100
5	0.0555	0.0491	0.2255	0.2213	0.0555	0.0529	-0.0033	-0.0026	-0.0067	-0.0065	-0.0033	-0.0030
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Obviously, if the coordinates of some joints are free, i.e. not specified, and only restricted to be inside of some interval, then the problem is non linear, because in the unknowns are included the values of x_i, y_i. In this case it is possible to extend the previous analysis using some type of trial and error procedure

EXTENSIONS OF THE METHOD

It is understood that the technique which has just been described can be extended to deal with more com plex cases which include conditions of slope and cur vature specified at one or various points of the polygon. The procedure is very simple and is exactly the same as for structural situations where boundary conditions are introduced automatically and in a general way for calculations to be carried out by computer. See reference [5] for an interesting description of this point.

Obviously, simultaneous treatment of the horizontal plane and elevation can be carried out following li nes parallel to those indicated. In such a case the functional to be minimized may be the following:

$$\int\left(\frac{1}{R^2} + \lambda^2 \frac{1}{T^2}\right)ds \qquad (12)$$

where 1/R and 1/T correspond to the mainly bending and torsional curvatures, and λ a parameter which can be specified according to the conditions of use of the road. The previous functional (12) may be, and is probably more adequately decomposed into the following:

$$\int\left(\frac{1}{R_H^2} + \frac{\lambda_1^2}{R_V^2} + \frac{\lambda_2^2}{T^2}\right)ds$$

with R_H and R_V the curvatures in the horizontal and vertical planes of the curve of the axis.

Finally, although the conexion with the planning of roads is less obvious, the method can be used succe ssfully along the lines indicated above for the representation of surfaces (e.g. land representation), surfaces which passing through a series of points $P_i(x_i,y_i,z_i)$ are found to be conditioned by simple continuity requierements, including the first deri vative or even the curvature. The most important aspect of the FEM both for this type of problem and for other problems, involves the selection of shape or interpolation functions, since the compo sing and solving of the system of equations (11) is standard and can be found in any general matrix programmes of structures, for example: SAP, STRUDL, ANSYS, NASTRAN, etc. Figure 8 shows the possibility of C^2 element which might be used for this type of problem of smooth and continous surface representation. If only the continuity of the slope is required, the selection of triangular and compati ble elements is very wide, and these elements correspond to all the compatible elements of bending of plates (refer to the specialised literature above all two recent publications [6] and [7].

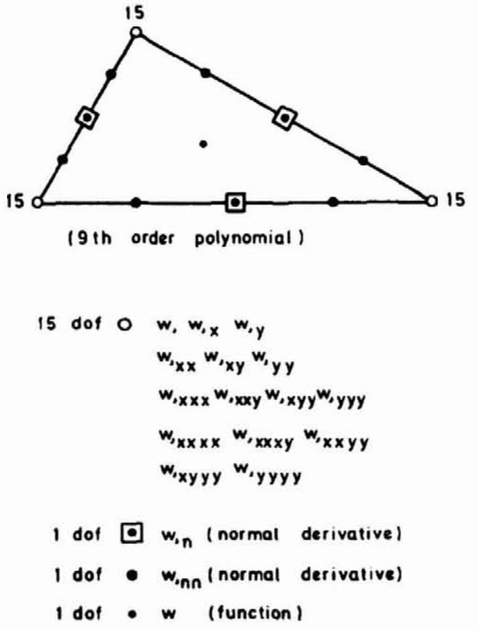

Figure 8. Continuity C^2 triangular element.

CONCLUSIONS

The FEM is and important tool to transform continuous problems into discret ones in Structural Analysis. Appliyng the method to the solving of problems in others fileds of Science and Technology has produced impressive results. The present work has shown by means of a simple example applied to road desing that the FEM offers significant possibilities in the solution of practical curve and surface fitting problems.

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